Partial semigroup actions and groupoids

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May 12, 2014
PARS

(joint work with Dana Williams)

Introduction

A few years ago, I was recruited by an Australian team to help them to show that the C*-algebras of their topological higher rank graph C*-algebras were nuclear.

It is well known that for a locally compact groupoid G with Haar system,

$$G \quad \text{amenable} \quad \Rightarrow \quad C^*(G) \quad \text{nuclear}$$

Therefore, the proof can be decomposed into two steps.

a) write the C*-algebra as a groupoid C*-algebra $C^*(G)$;

b) show that the groupoid G is amenable.

Deaconu-Renault groupoids

It is known that graph C*-algebras can be described as C*-algebras of groupoids of the following form:

Let X be a topological space and T a local homeomorphism from an open subset $\mathrm{dom}(T)$ of X onto an open subset $\mathrm{ran}(T)$ of X. Then

$$G(X,\mathbb{N},T) = \{(x, m-n, y) : m,n \in \mathbb{N}, T^m x = T^n y\}$$

has a natural étale groupoid structure.

This groupoid is often called a Deaconu-Renault groupoid but a better name is semigroup semi-direct product. A suggestive notation is

$$G(X,\mathbb{N},T) = X \rtimes_T \mathbb{N}$$

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The canonical cocycle

An essential feature of the semi-direct product is its canonical cocycle

$$c : G(X, \mathbb{N}, T) \to \mathbb{Z}$$

given by $c(x, m - n, y) = m - n$.
Therefore, $G(X, \mathbb{N}, T)$ can be viewed as an extension of the groupoid $c^{-1}(0)$ by the range of c.

One expects that the amenability of $c^{-1}(0)$ and the amenability of $\mathbb{Z}$ imply the amenability of $G(X, \mathbb{N}, T)$. A precise statement will be given later.

semigroup action by partial local homeomorphisms

Definition

A right action of a semigroup P on a topological space X is a map

$$(x, n) \in X * P \quad \mapsto \quad xn \in X$$

where $X * P$ is an open subset of $X \times P$, such that

1. for all $x \in X$, $(x, e) \in X * P$ and $xe = x$;
2. if $(x, m) \in X * P$, then $(xm, n) \in X * P$ iff $(x, mn) \in X * P$; if this holds, we have $(xm)n = x(mn)$;
3. for all $n \in P$, the map defined by $T_n x = xn$ is a local homeomorphism with domain $U(n) = \{x \in X : (x, n) \in X * P\}$ and range $V(n) = \{xn : (x, n) \in X * P\}$.

caveat

• One does not assume that P acts by homeomorphisms. The maps $T_n : U(n) \to V(n)$ are local homeomorphisms. One-sided subshifts of finite type or $z \mapsto z^2$ on the circle are such maps.

• One does not assume that the maps T_n are defined everywhere nor that they are surjective. Thus, we have a partial action of the semigroup P on X. We shall see that higher rank graphs lead to such semigroup actions. Another example was given by I. Putnam long time ago: start with a self-homeomorphism T of X and consider its restriction $T_{|U}$ where U is an open subset of X.

directed actions

Definition

Let us say that a semigroup action (X, P, T) is directed if for all pairs $(m, n) \in P \times P$ such that $U(m) \cap U(n)$ in non-empty, there exists $r = ma = nb$ such that $U(r) \supset U(m) \cap U(n)$.

Note that if T is everywhere defined, our condition says that P is directed with respect to the (left invariant) order relation $m \leq m'$ iff there exists $a \in P$ such that $m' = ma$. We shall only consider sub-semigroups $P \subset Q$ of a group Q. Then, the condition can be expressed as $P^{-1}P \subset PP^{-1}$. This condition is realized if $PP^{-1} = Q$. A semigroup which satisfies this condition is called a right reversible Ore semigroup.

well-directed actions

Definition

Let us say that a semigroup action (X, P, T) is well-directed if it is directed: for all pairs $(m, n) \in P \times P$ such that $U(m) \cap U(n)$ in non-empty, there exists $r = ma = nb$ such that $U(r) \supset U(m) \cap U(n)$ and if moreover $m, n \leq N$, where $N \in P$, one can find $r \leq N$.

semi-direct product

The following semi-direct groupoid appears in [Exel-R, Semigroups of local homeomorphisms and interaction groups, 2007] in the case when for all n, $U(n) = V(n) = X$.

Proposition

Let (X, P, T) be a directed semigroup action. Assume that P is a subsemi-group of a group Q. Then

$$G(X, P, T) = \{(x, mn^{-1}, y) \in X \times Q \times X : xm = yn\}$$

is a subgroupoid of $X \times Q \times X$ which carries an étale groupoid topology and a continuous cocycle $c : G(X, P, T) \to Q$ given by

$$c(x, mn^{-1}, y) = mn^{-1}.$$

topological higher rank graphs

Definition

A topological higher-rank graph graded by a semigroup P, or P-graph for short, is a topological small category Λ endowed with a map, called the degree map, $d : \Lambda \to P$ which satisfies the following properties

1. for all $m \in P$, $\Lambda^m = d^{-1}(m)$ is open;
2. for all $(\mu, \nu) \in \Lambda^{(2)}$, $d(\mu\nu) = d(\mu)d(\nu)$ and for all $v \in \Lambda^{(0)}$, $d(v) = e$;
3. it has the unique factorization property: for all $m, n \in P$, the composition map $\Lambda^m * \Lambda^n \to \Lambda^{mn}$ is a homeomorphism.

We define on Λ the order $\mu \leq \mu'$ iff there exists $\nu \in \Lambda$ such that $\mu' = \mu\nu$.

from SGA to THRG

Let $T : X * P \to X$ be a semigroup action as above. Then $\Lambda = X * P$ has a natural structure of topological higher rank graph. It is given by $\Lambda^{(0)} = X$, the range and source maps $r, s : \Lambda \to X$ are respectively $r(x, n) = x$ and $s(x, n) = xn$. The composition of arrows is the usual concatenation of paths:

$$(x, m)(xm, n) = (x, mn).$$

The degree map $d : \Lambda \to P$ is simply $d(x, n) = n$.

We shall see conversely how, under suitable assumptions, one can go from THRG to SGA.

assumptions

We make the following assumptions about the semigroup P:

- P is a subsemigroup of a group Q;
- $P \cap P^{-1} = \{e\}$;
- $PP^{-1} = Q$;
- the segments $[a, b] = aP \cap bP^{-1}$ are finite;
- P is quasi-lattice ordered: as soon as $a, b \in P$ have a c.u.b., they have a least c.u.b. denoted $a \vee b$.

and the following assumption about Λ:

- Λ is compactly aligned, i.e. if A, B are compact subsets of Λ, so is $A \vee B$.

associated semigroup action

Topological higher rank graphs provide semigroup actions.

Proposition

Let Λ be a P-graph satisfying above assumptions. Define

$$\Lambda * P = \{(\lambda, m) \in \Lambda \times P : m \leq d(\lambda)\}$$

and $T : \Lambda * P \to \Lambda$ *by* $T(\lambda, m) = \lambda m =: \nu$ *if* $d(\lambda) = mn$ *and* $\lambda = \mu\nu$ *with* $d(\mu) = m$ *and* $d(\nu) = n$.

1. *T is an action of P on Λ by partial local homeomorphisms.*
2. *The action is well-directed.*

order compactification

The action of P on Λ is not so interesting! it is proper, i.e. the semi-direct product $\Lambda \rtimes P$ is a proper groupoid. However the space Λ admits a natural compactification, its order compactification $\overline{\Lambda}$ which we are going to define and things become much more interesting!

For $\lambda \in \Lambda$, we define

$$F(\lambda) = \{\mu \in \Lambda : \mu \leq \lambda\}.$$

It is a closed subset of Λ. We define $\overline{\Lambda}$ as the closure of $F(\Lambda)$ with respect to Fell's topology in the space of closed subsets of Λ.

The elements of $\overline{\Lambda}$ can be viewed, equivalently, as paths (finite or infinite) or as hereditary and directed closed subsets of Λ.

the completed semigroup action

Proposition

Let Λ be a P-graph satisfying above assumptions. Then,

1. *the action of P on Λ extends to $\overline{\Lambda}$;*
2. *this action is by partial local homeomorphisms and it is well-directed;*
3. *the semi-direct groupoid $\overline{\Lambda} \rtimes P$ is étale, locally compact and Hausdorff.*

Remarks. 1) The higher rank C*-algebra is $C^*(\Lambda) = C^*(\overline{\Lambda} \rtimes P)$.
2) One also defines the boundary $\partial\Lambda$ of Λ, the boundary action and the associated groupoid $\partial\Lambda \rtimes P$.

amenability of a semi-direct product

Let us return to our initial goal: the amenability of $\overline{\Lambda} \rtimes P$ and of $\partial\Lambda \rtimes P$. It results from:

Theorem (R-Williams 2013)

Let (X, P, T) be a well-directed semi-group action where X is a locally compact Hausdorff space. Assume that P is a quasi-lattice ordered subsemi-group of a countable amenable group Q. Then the semi-direct product groupoid $G(X, P, T)$ is topologically amenable.

This is a corollary of the next theorem applied to the canonical cocycle $c : G(X, P, T) \to Q$. Since we assume that Q is amenable, it suffices to check that $c^{-1}(e)$ is amenable. This is true because it can be written as an increasing union of proper equivalence relations

$$R_n = \{(x, y) \in X \times X : \exists m \leq n : xm = ym\}$$

Our amenability result

Theorem (R-Williams 2013)

Let G be a locally compact groupoid with Haar system $\mathcal{G}$, Q a locally compact group and $c : G \to Q$ a continuous cocycle. Assume that Q and $c^{-1}(e)$ are amenable and that there exists a countable subset $D \subset Q$ such that

$$\forall x \in G^{(0)}, \quad c(G^x)D = Q,$$

then G is amenable.

Two previous results

Two particular cases were known:

Proposition (ADR 2000)

Let $c : G \to Q$ be a continuous cocycle. Assume that c is strongly surjective, i.e. $c(G^x) = Q$ for all $x \in G^{(0)}$. Then, the amenability of Q and of $c^{-1}(e)$ imply the amenability of G.

Proposition (Spielberg 2011)

Let $c : G \to Q$ continuous, where G is étale and Q is a countable discrete abelian group. Then the amenability of $c^{-1}(0)$ implies the amenability of G.

sketch of the proof

1. Borel and amenability coincide;
2. amenability is invariant under equivalence;
3. the amenability of the skew-product $G(c)$ implies the amenability of G;
4. the amenability of Q and of $c^{-1}(e)$ imply the amenability of the reduction $G(c)_{|Y}$ of $G(c)$ to the effective range of c:

$$Y = \{(r(\gamma), c(\gamma)) : \gamma \in G\} \subset G^{(0)} \times Q,$$

5. write $G^{(0)} \times Q$ as a countable union of translates Yq_i.

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